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Self-consistent hard-thermal-loop thermodynamics for the quark-gluon plasma ⋆

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Self-consistent approximations allowing the calculation of the entropy and the baryon density of a quark-gluon plasma are presented. These approximations incorporate the essential physics of the hard thermal loops, involve only ultraviolet-finite quantities, and are free from overcounting ambiguities. While being nonperturbative in the strong coupling constant g, agreement with ordinary perturbation theory is achieved up to and including order g^3. It is shown how the pressure can be reconstructed from the entropy and the baryon density taking into account the scale anomaly. The results obtained are in good agreement with available lattice data down to temperatures of about twice the critical temperature.

1 Introduction

While asymptotic freedom suggests that, at very high temperature and/or sufficiently large baryon density, QCD should behave as a weakly coupled quark-gluon plasma, lattice results reveal that this ideal gas limit is approached only rather slowly. Perturbation theory [1,2] fails to reproduce this behaviour correctly, and resummations beyond the conventional ring-resummation are needed to properly account for the thermodynamics. In QCD, the situation may not be hopelessly complicated however; in fact, the available lattice data are quite well reproduced by simple models based on non-interacting "quasiparticles" with temperature dependent masses [3,4], which suggests that it

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could be possible to describe the thermodynamics of the quark-gluon plasma in terms of its elementary excitations.

However, phenomenological fits using massive quasiparticles represent only a convenient parametrisation of the thermodynamics and further theoretical effort is needed to arrive at a complete understanding in terms of the elementary degrees of freedom and their interactions. What the quasiparticle picture suggests is that much of the physics at high temperature can presumably be described by propagator renormalisation or, in other words, a *resummed* perturbation theory in terms of *dressed* propagators. The importance of retaining their effects nonperturbatively has been demonstrated recently in scalar toy models [5,6]. The general framework for carrying out consistently such renormalisations of the thermodynamical functions has been well developed in the past in the context of the non-relativistic many-body problem [7–9].

A convenient quantity to consider when doing propagator renormalisation is the *entropy*, rather than the pressure [9,10]. Entropy is better suited, indeed, because it has a more direct interpretation in terms of quasiparticles; in particular, it is expressible entirely in terms of dressed propagators, at least in lowest orders in a skeleton expansion. Building upon such an expansion, we are able to construct approximations which incorporate in a consistent fashion all the important physics of the dominant contributions to the self-energies at high temperature, the so-called "hard thermal loops" (HTL) [11,12]. (Corresponding vertex corrections, a priori needed to maintain gauge symmetry, turn out to be unimportant at the order of interest.) In contrast to the direct HTL resummation of the pressure recently proposed in Ref. [13] the resummation that we perform is free of overcounting ambiguities, and involves only UV-finite expressions; when compared to perturbation theory it agrees up to, and including, order g^3.

In Ref. [14], this strategy has been successfully applied to estimate the entropy $\mathcal{S}$ of the purely Yang-Mills plasma. In this Letter, we consider two important extensions. The first one is the inclusion of quarks which allows us to also treat plasmas with a finite chemical potential μ. The relevant formalism, presented in Secs. 2 and 3 below, leads to self-consistent approximations for both the entropy $\mathcal{S}$ and the baryon density $\mathcal{N}$. The second extension concerns the computation of thermodynamical quantities other than the entropy or the density. In Sec. 4, we show how to reconstruct the pressure P by a numerical integration of $\mathcal{S}$ and $\mathcal{N}$; because of the trace anomaly, this requires some nonperturbative input which we take from lattice simulations at $\mu = 0$. For all temperatures $T \gtrsim 2 \ldots 3T_c$, our results are in remarkable agreement with the corresponding lattice results whenever the latter are available. We also present results for the baryon density at $T = 0$ and large μ. Further possible extensions of our method are outlined in Sec. 5.

2 Self-consistent approximation to the entropy and quark density

The thermodynamic potential $\Omega = -PV$ of a field theory involving bosons and fermions, expressed as a functional of full propagators (D for bosons, S for fermions) has the form [7]

$$\begin{aligned}
\Omega[D,S] &= \tfrac{1}{2}T\,\mathrm{Tr}\,\log D^{-1} - \tfrac{1}{2}T\,\mathrm{Tr}\,\Pi D - T\,\mathrm{Tr}\,\log S^{-1} + T\,\mathrm{Tr}\,\Sigma S + T\Phi[D,S] \\
&= T\Phi[D,S] + \mathrm{tr}\int\frac{d^4k}{(2\pi)^4} n(\omega)\,\mathrm{Im}\,\Big[\log D^{-1}(\omega,k) - \Pi(\omega,k)D(\omega,k)\Big] \\
&\quad +2\,\mathrm{tr}\int\frac{d^4k}{(2\pi)^4} f(\omega)\,\mathrm{Im}\,\Big[\log S^{-1}(\omega,k) - \Sigma(\omega,k)S(\omega,k)\Big] \qquad (1)
\end{aligned}$$

where $\Phi[D,S]$ is the sum of 2-particle-irreducible "skeleton" diagrams, $n(\omega) = (\mathrm{e}^{\beta\omega}-1)^{-1}$, $f(\omega) = (\mathrm{e}^{\beta(\omega-\mu)}+1)^{-1}$, and $\beta = 1/T$. Here "tr" refers to all discrete labels, including colour and flavour when applicable.

The self-energies $\Pi = D^{-1} - D_0^{-1}$ and $\Sigma = S^{-1} - S_0^{-1}$, where D_0 and S_0 are bare propagators, are themselves functionals of the full propagators and are determined by

$$\delta\Phi[D,S]/\delta D = \tfrac{1}{2}\Pi, \quad \delta\Phi[D,S]/\delta S = \Sigma, \qquad (2)$$

which goes hand in hand with the all-important stationarity property

$$\delta\Omega[D,S]/\delta D = 0 = \delta\Omega[D,S]/\delta S. \qquad (3)$$

A "self-consistent" ("Φ-derivable") [8] approximation is one that preserves this stationarity property by selecting a subset of skeleton-diagrams from Φ and determining the self-energies from (2). In particular, a two-loop approximation to $\Phi[D,S]$ (corresponding to a dressed one-loop approximation for the self-energies) is obtained by discarding all skeleton diagrams of loop-order 3 and higher.

This two-loop approximation for Ω has a remarkable consequence for the first derivatives of the thermodynamic potential, the entropy and the fermion densities:

$$\mathcal{S} = -\frac{\partial(\Omega/V)}{\partial T}\Big|_\mu, \quad \mathcal{N} = -\frac{\partial(\Omega/V)}{\partial \mu}\Big|_T. \qquad (4)$$

Because of the stationarity property (3), one can ignore the T and μ dependences implicit in the spectral densities of the full propagators, and differentiate exclusively the statistical distribution functions n and f in (1). Now the derivative of the *two-loop* functional $T\Phi[D,S]$ at fixed spectral densities of the propagators D and S turns out to just cancel that part of the terms $\mathrm{Im}\,(\Pi D)$ and $\mathrm{Im}\,(\Sigma S)$ in (1) which involves $\mathrm{Re}\,\Pi\,\mathrm{Im}\,D$ and $\mathrm{Re}\,\Sigma\,\mathrm{Im}\,S$, respectively. In a *self-consistent* two-loop approximation one therefore has the remarkably simple formulae

$$
\begin{aligned}
\mathcal{S} = & -\mathrm{tr}\int\frac{d^4k}{(2\pi)^4}\frac{\partial n(\omega)}{\partial T}\Big[\mathrm{Im}\,\log D^{-1}(\omega,k) - \mathrm{Im}\,\Pi(\omega,k)\,\mathrm{Re}\,D(\omega,k)\Big] \\
& -2\,\mathrm{tr}\int\frac{d^4k}{(2\pi)^4}\frac{\partial f(\omega)}{\partial T}\Big[\mathrm{Im}\,\log S^{-1}(\omega,k) - \mathrm{Im}\,\Sigma(\omega,k)\,\mathrm{Re}\,S(\omega,k)\Big]\,, \quad (5) \\
\mathcal{N} = & -2\,\mathrm{tr}\int\frac{d^4k}{(2\pi)^4}\frac{\partial f(\omega)}{\partial \mu}\Big[\mathrm{Im}\,\log S^{-1}(\omega,k) - \mathrm{Im}\,\Sigma(\omega,k)\,\mathrm{Re}\,S(\omega,k)\Big]\,. \quad (6)
\end{aligned}
$$

This has been noted first for the entropy in a non-relativistic context by Riedel [10] and more recently in QED by Vanderheyden and Baym [15], but this important simplification holds much more generally and is equally valid for the fermion density [16].

In a nonabelian gauge theory, the above relations have to be in general augmented by ghost contributions, which can however be avoided in those gauges where ghosts do not propagate, such as Coulomb gauge, in which we shall work in what follows. At any rate, in gauge theories, abelian as well as nonabelian, self-consistency does not guarantee gauge invariance, for only propagators have been dressed, and no vertices [1] .

However, in what follows we shall consider *approximately self-consistent* approximations which guarantee gauge independence. These amount to compute the self-energies in (resummed) perturbation theory, satisfying the self-consistency condition in a perturbative sense. On the other hand, we shall not expand out the self-energies from eqs. (5) and (6) so that the resulting approximations for the $\mathcal{S}$ and $\mathcal{N}$ remain *nonperturbative* in g. This is possible without encountering ultraviolet divergences because (in contrast to the original thermodynamic potential) the expressions (5) and (6) have the decisive advantage of being manifestly *ultraviolet finite* since the derivatives of the distribution functions vanish exponentially for both $\omega \to \pm\infty$. Moreover, any multiplicative renormalization with real Z leaves (5) and (6) unchanged.

[1] Vertices can be dressed in a self-consistent manner in the formalism worked out in Ref. [17].

3 Contact with perturbation theory

In Coulomb gauge, the gluon propagator consists of a transverse and a longitudinal piece, with LO self-energies at soft momenta ($\omega, k \ll \max(T, \mu)$) given by the so-called hard thermal (dense) loops [11,12]

$$\hat{\Pi}_L(\omega, k) = \hat{m}_D^2 \left[1 - \frac{\omega}{2k} \log \frac{\omega + k}{\omega - k}\right],$$
$$\hat{\Pi}_T(\omega, k) = \frac{1}{2} \left[\hat{m}_D^2 + \frac{\omega^2 - k^2}{k^2}\, \hat{\Pi}_L\right], \tag{7}$$

where

$$\hat{m}_D^2 = (2N + N_f)\, \frac{g^2 T^2}{6} + N_f\, \frac{g^2 \mu^2}{2\pi^2} \tag{8}$$

is the (leading-order) Debye screening mass. [2] As is well known [18], the gluon propagator dressed by these self-energies has quasiparticle poles at $\omega_{T,L}(k)$ with momentum-dependent effective masses and Landau damping cuts for $|\omega| < k$. When $k \gg \hat{m}_D$, the pole corresponding to the collective longitudinal excitation has exponentially vanishing residue [19], whereas that of the transverse excitations tend to $\sqrt{k^2 + m_\infty^2}$, where the asymptotic mass is given by

$$m_\infty^2 = \hat{\Pi}_T(k, k) = \tfrac{1}{2} \hat{m}_D^2 \,. \tag{9}$$

The (massless) quark propagator at finite temperature or density is split into two separate branches of opposite ratio of chirality over helicity with propagators $\Delta_\pm = [-\omega + k \pm \Sigma_\pm]^{-1}$. In the HTL approximation, the respective self-energies read [18,12]:

$$\hat{\Sigma}_\pm(\omega, k) = \frac{\hat{M}^2}{k} \left(1 - \frac{\omega \mp k}{2k} \log \frac{\omega + k}{\omega - k}\right), \tag{10}$$

where ($C_f = (N^2 - 1)/2N$)

$$\hat{M}^2 = \frac{g^2 C_f}{8} \left(T^2 + \frac{\mu^2}{\pi^2}\right). \tag{11}$$

[2] For simplicity we shall write out our formulae for only one value of the chemical potential; the generalization to several different chemical potentials is straightforward.

The dressed fermion propagators have quasiparticle poles at $\omega_\pm(k)$ and Landau damping cuts. At large k and positive frequency $\omega_+ \to \sqrt{k^2+M_\infty^2}$ with asymptotic mass $M_\infty^2 = 2\hat{M}^2$; the abnormal branch ω_- has exponentially vanishing residue [19].

Although the various HTL self-energies constitute a leading-order result only under the condition of *soft* momenta and frequencies, the result for the asymptotic masses applies also for *hard* momenta [20].

The LO interaction contribution $\propto g^2$ in our expressions for the entropy and the density arise from the domain of hard momenta. Because of the subtraction terms in (5) and (6), it turns out [14,16] that this contribution is given entirely in terms of the LO values for $\mathrm{Re}\,\Pi_T(\omega^2=k^2)$ and $\mathrm{Re}\,\Sigma_\pm(\omega=\pm k)$, yielding

$$\mathcal{S}^{(2)} = -N_g \frac{m_\infty^2 T}{6} - NN_f \frac{M_\infty^2 T}{6}, \quad \mathcal{N}^{(2)} = -NN_f \frac{M_\infty^2 \mu}{2\pi^2} \tag{12}$$

($N_g = N^2-1$) in agreement with known results [1].[3] Notice the simplicity of these results in comparison with conventional two-loop calculations. Quite generally the LO interaction results for entropy and fermion density are given by $\mathcal{S}^{(2)} = -T\{\frac{1}{12}\sum_B m_{\infty\,B}^2 + \frac{1}{24}\sum_F M_{\infty\,F}^2\}$, $\mathcal{N}^{(2)} = -\frac{1}{8\pi^2}\sum_F \mu_F M_{\infty\,F}^2$ where the sums run over all bosonic (B) and fermionic (F) degrees of freedom. Thus the LO interaction terms in $\mathcal{S}$ and $\mathcal{N}$ are entirely given by the quasiparticle dispersion relations at large momenta. In fact, these formulae for $\mathcal{S}^{(2)}$ and $\mathcal{N}^{(2)}$ are identical to those corresponding to massive non-interacting particles (with constant masses $m = m_\infty$ and M_∞, respectively) when expanded to leading order in m^2.

When $T \gg \hat{m}_D$, the NLO contributions are of order g^3 and in the conventional perturbation theory for the pressure arise from the sum of (electro-static) ring diagrams,[4] which yields $P^{(3)} = \frac{1}{12\pi} N_g \hat{m}_D^3 T \propto g^3$.

If the propagators and self-energies in (5) and (6) are those of the HTL approximation, the order g^3 contributions in the corresponding expressions for $\mathcal{S}$ and $\mathcal{N}$, denoted $\mathcal{S}_{\mathrm{HTL}}$ and $\mathcal{N}_{\mathrm{HTL}}$, turn out to give only the part of the full result that is obtained by differentiating $P^{(3)}$ at fixed $\hat{m}_D$ [14,16], $\mathcal{N}_{\mathrm{HTL}}^{(3)} = 0$ and $\mathcal{S}_{\mathrm{HTL}}^{(3)} = \frac{1}{12\pi} N_g \hat{m}_D^3$. The remaining contributions of order g^3 come from NLO corrections to the asymptotic quasiparticle mass and are calculable in HTL perturbation theory [16]. These corrections are, in contrast to the LO asymptotic masses, momentum-dependent, and increasing at hard momenta. In order to estimate their contribution in our numerical evaluations, we shall

[3] By contrast, the direct HTL resummation of the *pressure* overincludes the LO interaction contributions $\propto g^2$ [13,21].

[4] If $T \lesssim g\mu$, the ring diagrams give rise to contributions of order $g^4 \log(g)$.

approximate them by averaged, effective NLO corrections to m_∞ and M_∞ which are uniquely determined through eq. (12) as [16]

$$\bar\delta m_\infty^2 = -\frac{1}{2\pi} g^2 N T \hat m_D, \quad \bar\delta M_\infty^2 = -\frac{1}{2\pi} g^2 C_f T \hat m_D. \tag{13}$$

In these expressions, the linear $\hat m_D$ dependence and the Casimir factors are indeed those expected from the corresponding HTL-resummed one-loop diagrams.

4 Nonperturbative evaluation — recovering the pressure

In our nonperturbative, numerical evaluation of (5) and (6) we consider two successive approximations. The first consists of using HTL-resummed propagators, which as we have discussed reproduces the LO interaction terms and part of the NLO one. As a second step we take into account the NLO corrections to the asymptotic mass in their averaged form (13) and we do so by $m_\infty^2 \to \bar m_\infty^2 = m_\infty^2[1+\bar\delta m_\infty^2/m_\infty^2]^{-1}$ and similarly for the fermionic asymptotic mass M_∞. This simple Padé resummation can be shown to give a surprisingly good approximation to the gap equation in exactly solvable scalar models [16]. However, because (13) refers to the asymptotic mass and because the plasma frequency and the Debye mass turn out to have rather different NLO corrections [22–24], we restrict this modification to hard momenta defined by $k > \Lambda = \sqrt{2\pi T \hat m_D c_\Lambda}$ and vary c_Λ to test the stability of the final results.

$\mathcal{S}_{\rm HTL}$ can be separated in two physically distinct contributions: one from quasiparticle poles, the other from Landau damping cuts of the various excitations. The gluonic ones, which only contribute to the entropy, have been given in eqs. (16) and (17) in Ref. [14]. The fermionic ones are quite analogous and read $\mathcal{S}_{f\,\rm HTL} = \mathcal{S}_{f\,\rm HTL}^{\rm QP} + \mathcal{S}_{f\,\rm HTL}^{\rm LD}$ with

$$\begin{aligned}\mathcal{S}_{f\,\rm HTL}^{\rm QP} = 2NN_f \int \frac{d^3k}{(2\pi)^3} \frac{\partial}{\partial T} \Big\{ & T \log(1+{\rm e}^{-[\omega_+(k)-\mu]/T}) \\ & + T \log \frac{1+{\rm e}^{-[\omega_-(k)-\mu]/T}}{1+{\rm e}^{-(k-\mu)/T}} + (\mu \to -\mu) \Big\}\end{aligned} \tag{14}$$

where only the explicit T dependences are differentiated and not those implicit in the dispersion laws $\omega_+(k)$ and $\omega_-(k)$ of the fermionic quasiparticles.

The contribution to the quark density is obtained by replacing $\partial/\partial T$ in the above formula by $\partial/\partial\mu$. In the limit $T \to 0$, the resulting expression can be simplified to read ($\mu > 0$)

$$\mathcal{N}_{f\,\mathrm{HTL}}^{\mathrm{QP}}\Big|_{T=0} = N N_f \int\limits_0^\mu \frac{k^2 dk}{\pi^2} [\theta(\mu - \omega_+(k)) - \theta(\omega_-(k) - \mu)]. \tag{15}$$

The fermionic Landau-damping contribution to the entropy is

$$\begin{aligned}\mathcal{S}_{f\,\mathrm{HTL}}^{\mathrm{LD}} = -4 N N_f \int \frac{d\omega\, d^3k}{(2\pi)^4} \frac{\partial f(\omega)}{\partial T} \theta(k^2 - \omega^2) \Big\{ &\arg[k - \omega + \Sigma_+(\omega, k)] \\ &- \mathrm{Im}\,\Sigma_+(\omega, k)\, \mathrm{Re}\,[k - \omega + \Sigma_+(\omega, k)]^{-1} \\ + \arg[k + \omega + \Sigma_-(\omega, k)] - \mathrm{Im}\,\Sigma_-(\omega, k)\, \mathrm{Re}\,[k + \omega + \Sigma_-(\omega, k)]^{-1} \Big\}\end{aligned} \tag{16}$$

and that to the quark density, $\mathcal{N}_{f\,\mathrm{HTL}}^{\mathrm{LD}}$, is again obtained by replacing $\partial/\partial T$ by $\partial/\partial\mu$. In the limit of zero temperature we simply have $\partial f/\partial\mu \to \delta(\omega - \mu)$.

The coupling constant in the HTL masses $\hat{m}_D$ and $\hat{M}$ is determined from the two-loop renormalization group equation and as a central value for the ($\overline{\mathrm{MS}}$) renormalization scale $\bar{\mu}$ we choose for the case of zero chemical potential the spacing of the Matsubara frequencies $2\pi T$, and for the case of zero temperature the diameter of the Fermi sphere, 2μ. The QCD scale $\Lambda_{\overline{\mathrm{MS}}}$ is, according to lattice data, rather close to T_c at $\mu = 0$; for definiteness we adopt [5] $T_c/\Lambda_{\overline{\mathrm{MS}}} = 1.14$, a recent lattice result [25] for the case of pure glue QCD, which we take also for $N_f \neq 0$ because lattice data so far showed little sensitivity of this parameter on N_f. In order to have an estimate of the theoretical uncertainty of our approach we consider a variation of $\bar{\mu}$ around the central value by a factor of 2.

Were it not for the trace anomaly [26] $\langle T_\mu^\mu \rangle = \mathcal{E} - 3P \neq 0$ the pressure would be simply given by $P = (T\mathcal{S} + \mu\mathcal{N})/4$ (in the case of massless quarks). The correct relation is instead provided by an integration such as

$$P(T, \mu) = \int\limits_{T_1}^{T} \mathcal{S}(T', \mu)\, dT' + P(T_1, \mu)\,. \tag{17}$$

At $\mu = 0$, the integration constant can be taken from the lattice; for definiteness we take $T_1 = T_c$ and put $P(T_c, 0) = 0$. (Nonzero values as given e.g. by the lattice results do not modify the following results significantly for $T > T_c$.)

In Fig. 1 our results for the entropy (a) and the pressure (b) of pure-glue QCD

[5] This value is about 10 % higher then the one used in Refs. [13,14]; the consequences for the following results are however rather small.

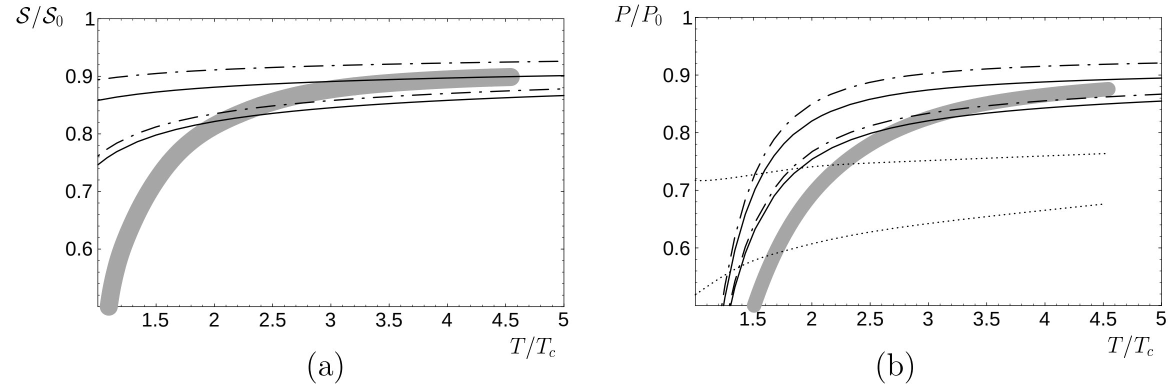

Fig. 1. Comparison of our results (full lines LO; dash-dotted lines: NLO) for the entropy density (a) and the pressure (b) of a pure gluon plasma with the lattice results (grey bands) from Ref. [27]. See text for detailed explanations.

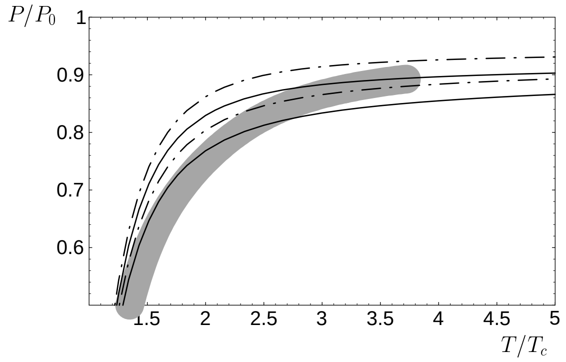

Fig. 2. Comparison of our results for the pressure for a quark-gluon plasma with two massless flavours with a continuum extrapolation of recent lattice results from Ref. [29].

are compared with the lattice results of Ref. [27] [6]. Full lines give the upper and lower bounds of the result in the HTL approximation under a variation of the renormalization scale as described above, the dash-dotted lines the corresponding one for the NLO approximation, which include a variation of c_Λ in the range $\frac{1}{2} \ldots 2$. The lattice results are given by grey bands with the thickness giving a typical error. For comparison, the result of a direct HTL-resummation of the pressure as reported in Ref. [13], which fails to incorporate the correct LO term but reproduces the order g^3-part, is given by the dotted line in Fig. 1b (transcribed to our value of $T_c/\Lambda_{\overline{\rm MS}} = 1.14$).

Fig. 2 gives our results for the pressure in the presence of two massless quark flavours and compares with the (estimated) continuum extrapolation of recent lattice results [29]. The remarkably accurate agreement with the lattice result already at $T > 2 \ldots 3T_c$ is seen to persist also in the presence of fermions.

[6] The more recent lattice results obtained with a RG-improved lattice action in Ref. [28] are consistent with the results of Ref. [27] within the given errors, but are systematically higher by some 3–4 %.

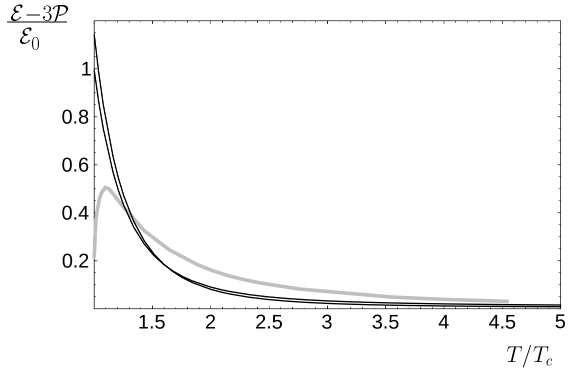

Fig. 3. Comparison of our results for the trace anomaly in a pure gluon plasma with the lattice result from Ref. [27].

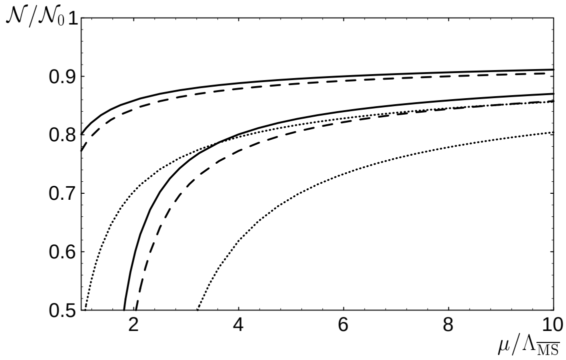

Fig. 4. Our result for the baryon density for $N_f = 3$ in comparison with order-g^2 (dashed line) and order-g^4 (dotted line) perturbation theory.

In Fig. 3 the result for the trace anomaly is shown for the case of pure-glue QCD, in comparison with the lattice data from Ref. [27]. Here our results are very weakly dependent on $\bar{\mu}$ and also LO and NLO results almost coincide; in fact, at small T our results are dominated by the integration constant $P(T_c, 0)$. Evidently, the details very close to T_c are not reproduced, but a substantial part of the lattice result is seen to be accounted for in our approach.

Note that it is not possible to determine a particular $P(T_1)$ from the requirement that both $p(\alpha_s) = P(T)/P_0(T)$ and $s(\alpha_s) = S(T)/S_0(T)$ approach 1 in the limit of $\alpha_s \to 0$, because the differential equation $p(\alpha_s) + \frac{1}{4}\beta(\alpha_s)p'(\alpha_s) = s(\alpha_s)$ with $\beta(\alpha_s) = -\beta_0\alpha_s^2 - \beta_1\alpha_s^3 - \ldots$ has as homogeneous solutions

$$p(\alpha_s)_{\mathrm{hom.}} = C\,\mathrm{e}^{-\frac{1}{\alpha_s}[4\beta_0^{-1}+O(\alpha_s)]},$$

and this vanishes at $\alpha_s = 0$ together with all its derivatives! Different $P(T_c, 0)$ thus correspond to exactly the same coefficients of a series expansion of p in α_s. Because it does not alter the entropy, the physical interpretation of this additional input is obviously that of fixing a (strictly nonperturbative) bag constant.

At zero temperature we are lacking the information on the value of a critical chemical potential μ_c and of $P(\mu_c)$. However, we can still calculate $\mathcal{N}$, and the result of a numerical evaluation for $N = 3$, $N_f = 3$ is displayed in Fig. 4. (NLO corrections are not needed now because the g^3-contribution is zero and therefore the HTL result is trivially correct at this order, too.) Also given are the perturbative results to order g^2 and g^4 from Ref. [1,17] (translated to the $\overline{\text{MS}}$-scheme). A simple quasiparticle model with constant mass $m = M_\infty$ would result in values for $\mathcal{N}$ only slightly ($\sim$ 1%) above the perturbative g^2-result. Our results are still close to the latter, but somewhat higher. The perturbative order-g^4 results on the other hand are significantly lower with little overlap with the order-g^2 results. From the good agreement of our above results for the entropy with lattice results we expect that our approximations for the density are similarly stable for μ sufficiently above μ_c.

5 Outlook

In our numerical evaluations, we have included NLO effects through the averaged values (13), which suffices to restore the correct coefficient of g^3 contributions. A more complete treatment would involve the exact result for $\delta\Pi_T(\omega^2 = k^2)$ and $\delta\Sigma_\pm(\omega = \pm k)$ in HTL-resummed perturbation theory. Their numerical evaluation and inclusion in $\mathcal{S}$ and $\mathcal{N}$ is work in progress.

Another extension of the present results which is under way is the evaluation for general $\mu > 0$ and $T > 0$. There the Maxwell relations

$$\frac{\partial \mathcal{S}}{\partial \mu}\Big|_T = \frac{\partial \mathcal{N}}{\partial T}\Big|_\mu \tag{18}$$

are fulfilled up to and including order g^3 upon inclusion of the NLO contributions. At and beyond order g^4, (18) constitute a nontrivial constraint on the renormalization-group flow of $\alpha_s(T,\mu)$, determining $P(T,\mu)$ in terms of the initial data at $\mu = 0$, which can be taken from the lattice. This fixes the equation of state in the entire μ-T plane. A similar program has been carried out recently in Ref. [30] in simple quasiparticle models with momentum-independent transverse-gluon and quark masses. Within our approach, this can be extended to include both the effects of momentum-dependence of the thermal masses as well as Landau damping nonperturbatively, while maintaining equivalence with conventional perturbation theory up to and including order g^3 in the thermodynamic potentials.

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